Amplitude-independent pulse length determination using the Cepstrum analysis

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Date 15.03.1999

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1.0 Continuous calculation

The Fourier transform is an essential tool in engineering. The Fourier transform of a signal in the time domain s(t) is the continuous spectral function of the signal in the frequency domain S(f). The calculation is performed with

$$S(f) = \int_{-\infty}^{+\infty} s(t)\ e^{-i2\pi ft}\ dt$$

The transformation back into the time domain is carried out with

$$S(t) = \int_{-\infty}^{+\infty} s(f)\ e^{i2\pi ft}\ df$$

Cepstrum analysis is an analytical method based on the Fourier transform.

The calculation of the cepstrum of a signal s(t) from the time domain is carried out by complex logarithmization of the Fourier transform S(f) and subsequent inverse Fourier transform.

$$C(t) = \int_{-\infty}^{+\infty} \ln(s(f))\ e^{i2\pi ft}\ df$$

The Fourier transform of a real time signal is a complex function. Its logarithmization is limited to the main values.

$$\ln(x+iy) = \tfrac{1}{2} \ln(x + y^{22}) + i*[\mathrm{sgn}(y)*\pi/2 - \arctan(x/y)]$$

One application of cepstrum analysis is the detection of signal echoes, such as those that occur in multipath propagation [1].

The aim of this work is to show that the duration of pulses can be determined with the help of the cepstrum, independent of the signal amplitude in the time domain.

For example, if the time function of a pulse is

$$s(t) = a * \sin(2*\pi*f_0 *t) \quad 0 <= t <= \tau$$

gives the Fourier transform of s(t) with

$$S(f) = \int_0^{\tau} a*\sin(2*\pi*f_0*t)\ e^{-i2\pi ft}\ dt$$

and, after a short calculation, a form of the part of the spectrum above f_0 that is favorable for logarithmization .

$$S(f) = \frac{a * (f-f_0)}{2\pi(f^2 - f_0^2)} * [1 - e^{-i2\pi (f+f0)\tau}]$$

$$\ln(S(f)) = \ln\left(\frac{a *(f-f_0)}{2\pi(f^2 - f_0^2)}\right) + \ln[1 - e^{-i2\pi (f+f0)\tau}]$$

According to the properties of the Fourier transform, the summands can be transformed individually into the time domain. The same calculation can be performed for the part of the spectrum below f_0.

It is to be expected that the duration of the pulse can be recognized in the cepstrum of s(t) because the pulse length is formally included. Because of its complex part, the second summand will be considered in the following.

The transformation into the time domain is carried out with

$$C(t) = \int_{-\infty}^{+\infty} \ln(1 - e^{-i2\pi (f+f0)\tau}) \ e^{i2\pi ft} df$$

Partial integration leads to considerable difficulties here, but the first factor in the integral can be represented by a series.

$$\ln(1-x) = \sum_{k=1}^{\infty} x^k / k \quad k \in N$$

By inserting you get

$$C(t) = \int_{-\infty}^{+\infty} \sum_{k=1}^{\infty} 1/k \; e^{i2\pi(f+f0)[f*t/(f0+f)-k*\tau]} \, df$$

$$C(t) = \sum_{k=1}^{\infty} 1/(k*g) \; \delta(t-k*\tau) \quad \forall k \in N$$

$\delta(t-\tau)$ is the delta function, with the property of being undefined at the point of its occurrence and zero otherwise. Its area is one [2],[3],[4]. It corresponds to a Needle impulse.

The needle pulses therefore occur in the cepstrum of s(t) at the times $n\tau$, which correspond to whole multiples of the pulse length of S(t).

In addition, the equation shows that the amplitude of the needle pulses is independent of the amplitude of s(t). The factor g only contains multiples of 2π and the pulse duration.

2.0 Discrete calculation

A sampling series consisting of M=16384 samples is populated to 1/8 of the number with values of the function

$$s(m) = a*\sin(2*\pi*m/M)$$

where a=32768 is initially assumed.

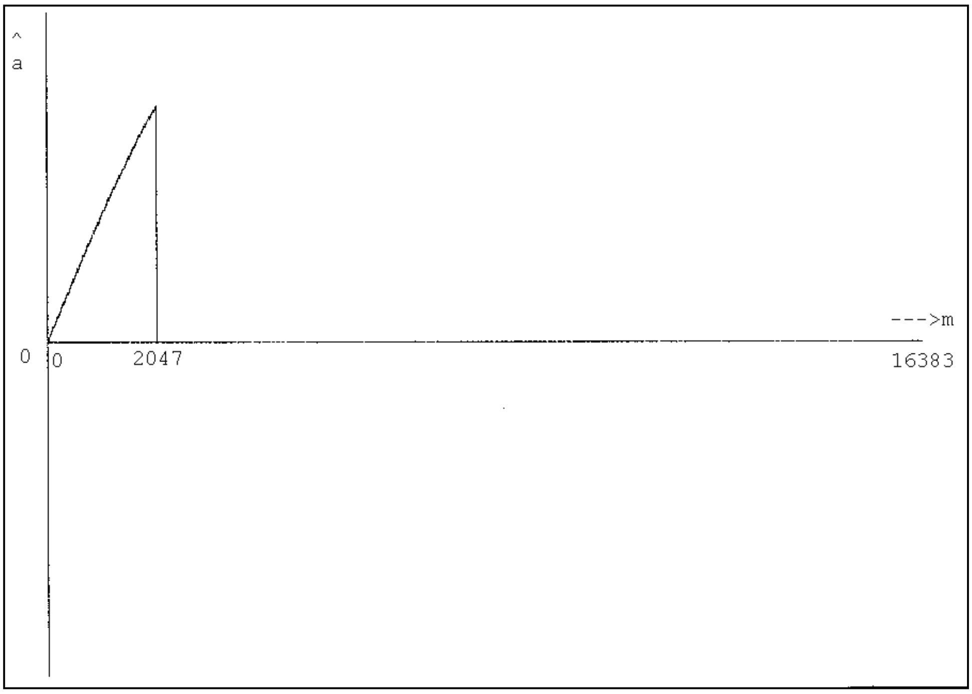

Figure 1 - Scanning series s(m)

To avoid spectral broadening in the discrete spectrum of s(m), the sampling series is evaluated with the Hanning function.

$$w(m) = ½(1 - \cos(2\pi m/M))$$

The transformation into the frequency range is carried out with

$$S(n) = \sum_{n=0}^{M-1} s(m)\ e^{-i2\pi mn/M}$$

A Fast Fourier transform calculates the discrete Fourier transform, which is now an approximation of the coefficients to the Fourier series.

Figure 2 shows the magnitude of the spectral lines for positive frequencies (0 <= n <= N/2) and negative frequencies (N/2 < n < N). As this is a real sampling series, the spectral values of the negative frequencies are conjugate-complex to those of the positive frequencies.

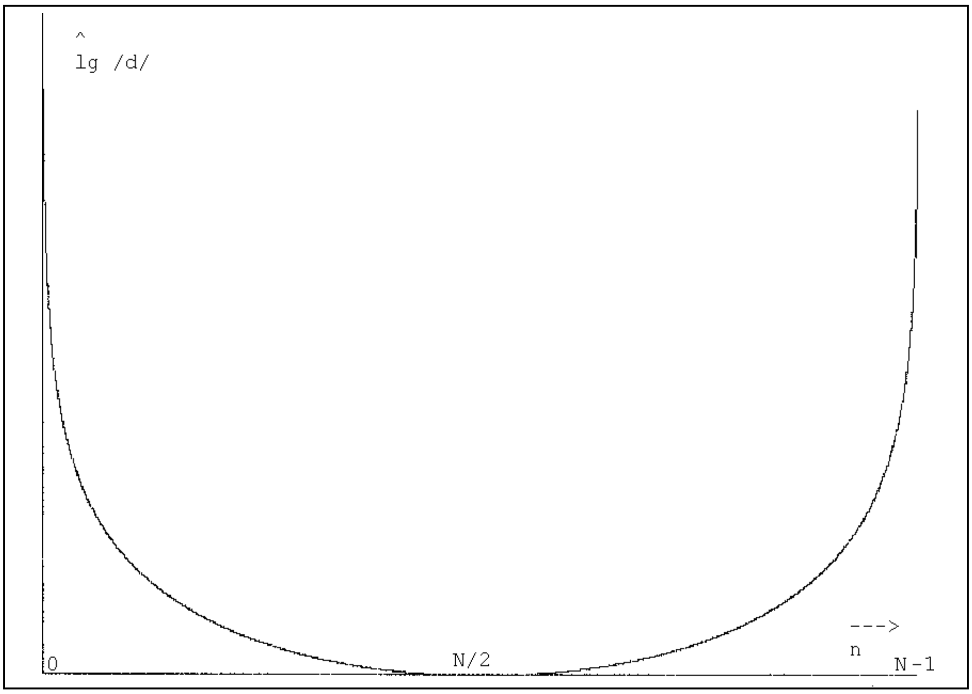

Figure 2 - Semi-logarithmic representation of the amount of the discrete spectrum

The complex logarithmization of the N spectral values is then performed.

An inverse Fast Fourier transform then calculates the cepstrum of s(m).

Figure 3 shows the magnitude of the cepstrum consisting of N cepstral values. In contrast to the discrete spectrum, the second half is not redundant to the first half of the data. However, the temporal arrangement of the delta functions, which appear as the sum of cos(x)/x + i*sin(x)/x due to the time-limited calculation, is symmetrical to n=N/2.

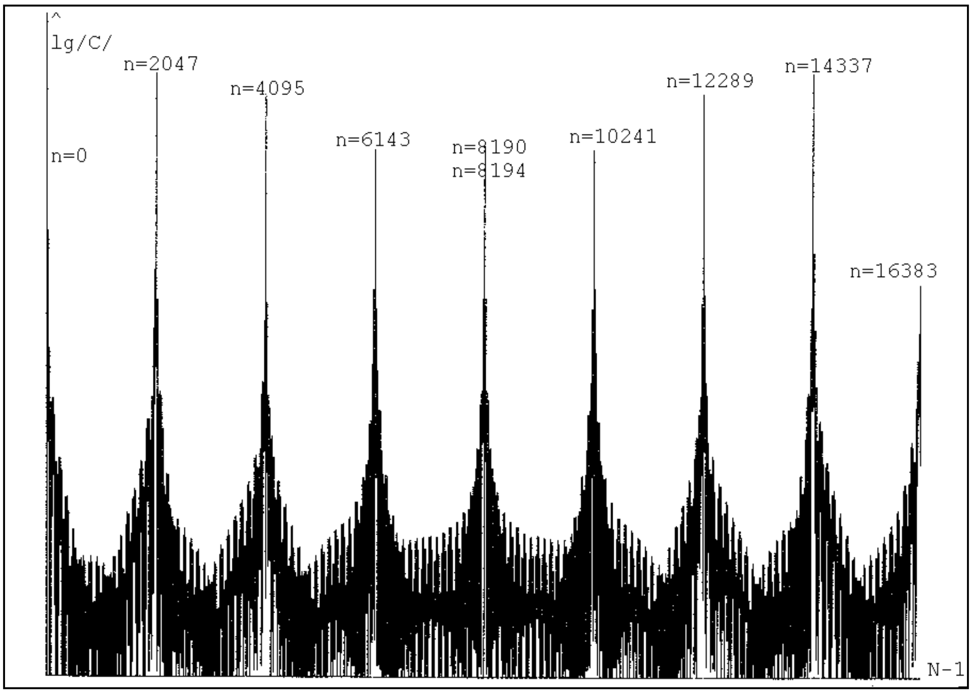

Figure 3 - Semi-logarithmic representation of the amount of the cepstrum of s(m)

The delta functions can be found at $N/2+k*m_\tau +1$ and $N/2-k*m_\tau -1$, where m_τ is 1/8 * M in this example. If T_A is the sampling interval, the pulse length results, among other things, in

$$\tau = (n_k - N/2 - 1) / k * T_A .$$

By transformation into the frequency domain, de-logarithmization, back-transformation into the time domain and inverse Hanning evaluation, the sampling series s(m) can be recovered exactly, as with the simple Fourier transformation.

In words, the occurrence of the delta functions can be formulated as follows:

> Delta functions occur at points in time at which an echo of s(t) delayed by τ and amplified by g overlaps it linearly **or** s(t) itself overlaps linearly amplified by g after the time τ, as in the example above **or** at other mathematically discontinuous points of s(t).

The question to what extent s(n) or s(t) can be reconstructed beyond τ by operations in the cepstrum is left aside here.

Incidentally, a coordinate transformation from rectangular to polar coordinates can also be used instead of the complex logarithmization. The subsequent inverse FFT results in the same temporal positioning and amplitude of the

Delta functions in the cepstrum, whereby a calculation with zeros in the spectrum is permitted.

Figure 4 shows the cepstrum of s(m) with a=1, which corresponds to an attenuation of 90.31 dB compared to the example above.

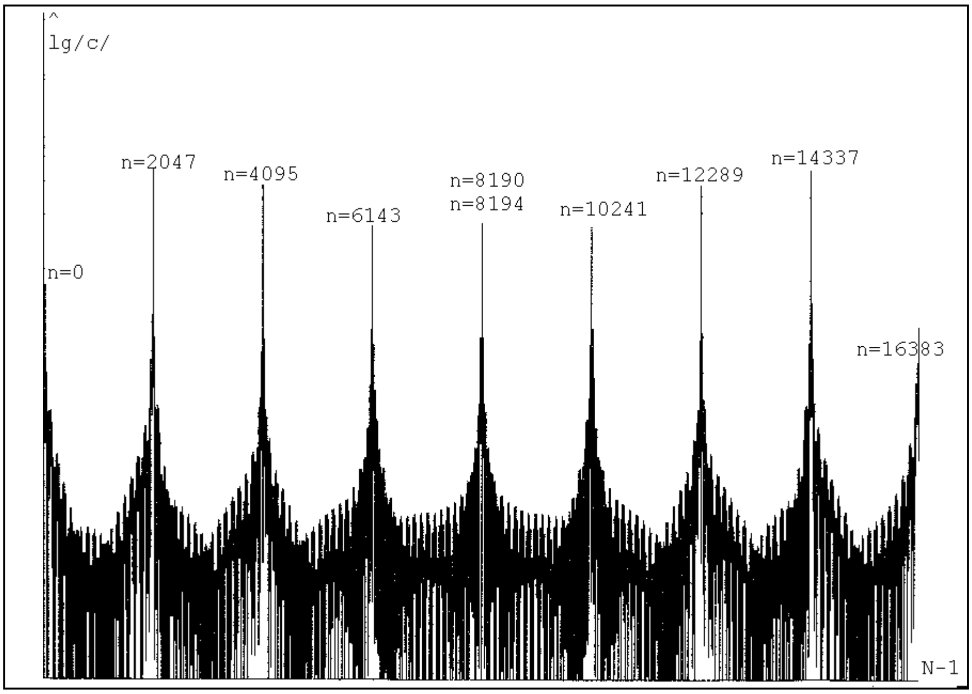

Figure 4 - Semi-logarithmic representation of the amount of the cepstrum of s(m) with s(m) = 1 * sin(2πm/M)

It is noteworthy that the amplitude of the delta functions and their temporal arrangement do not differ from those of the cepstrum of the sampling series s(m) = 32768 * sin(2πm/M).

3.0 Some examples

Figure 5 shows the magnitude of the complex cepstrum of a unipolar rectangular pulse over ¼ of the number of samples. The delta functions can be found at n+1 = ¼*N and n-1 = N - ¼*N. The amplitude of the sampling series is 32768.

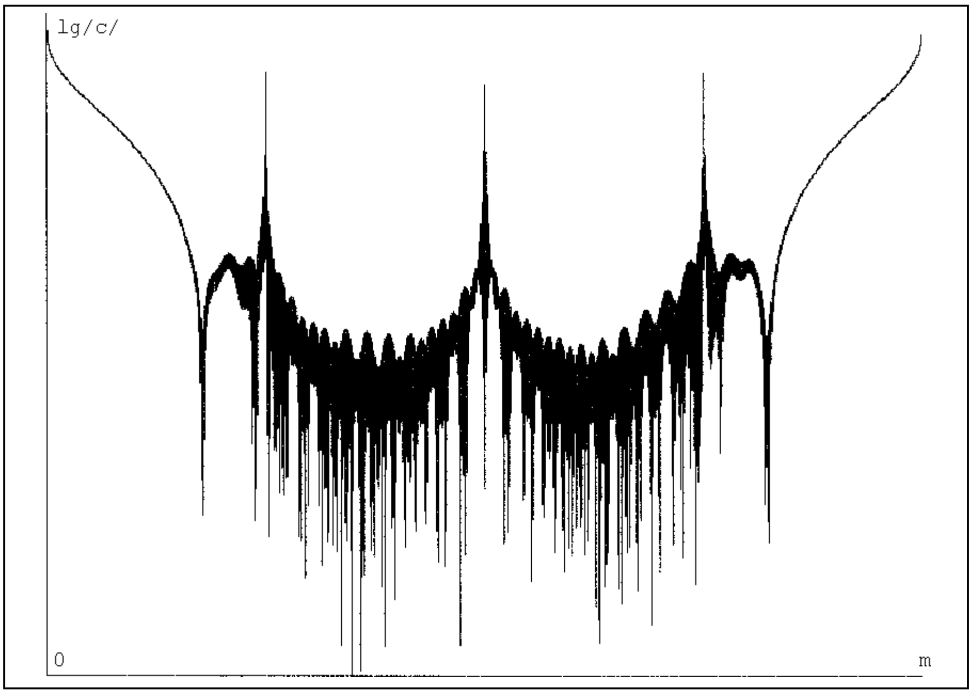

Figure 5 - Cepstrum, shown semi-logarithmically

Figure 6 is the magnitude of the complex cepstrum of a unipolar rectangular pulse over half the number of samples.

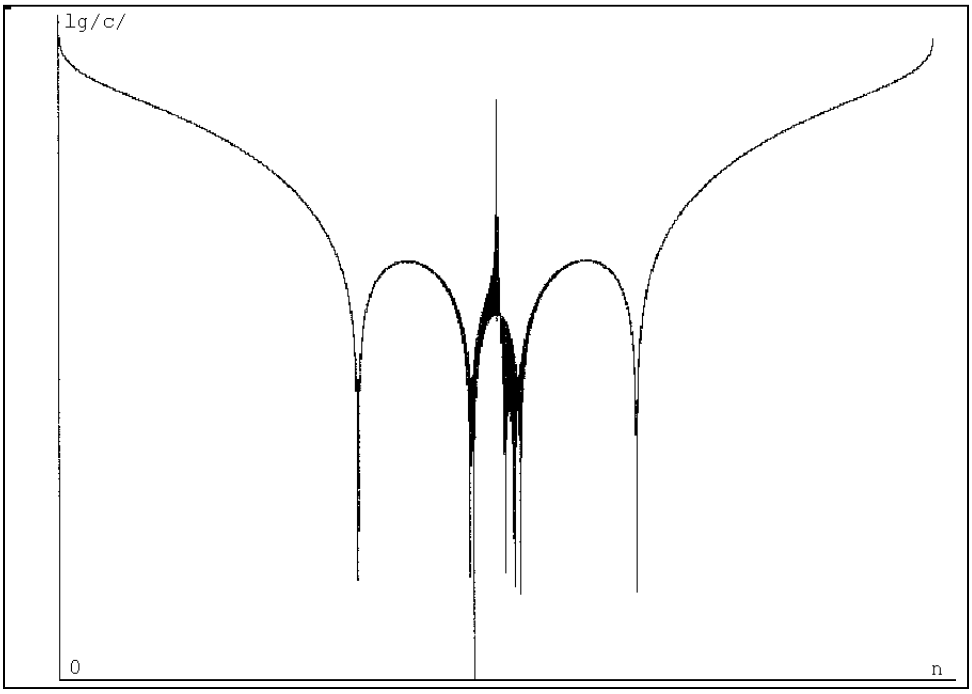

Figure 6 - Cepstrum, shown semi-logarithmically

Figure 7 is the complex cepstrum of a unipolar rectangular pulse over ¾ of the number of samples. The delta function with the maximum amplitude can be found at n-1 = 0.75 * N.

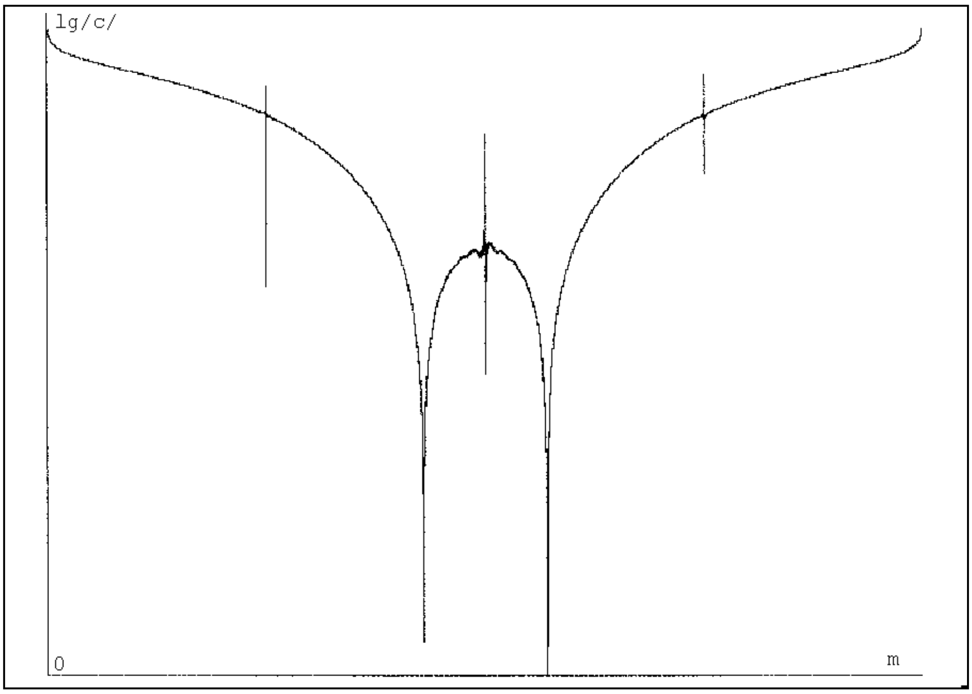

Figure 7 - Complex cepstrum, shown semi-logarithmically

Figure 8 shows the complex cepstrum of a unipolar rectangular pulse, which decreases linearly from m = 1/4 * M down to zero at m=1/2 * M. The delta pulse with the maximum amplitude is at n-1 = ¼ * M. Here, too, the amplitude of the pulse is 32768.

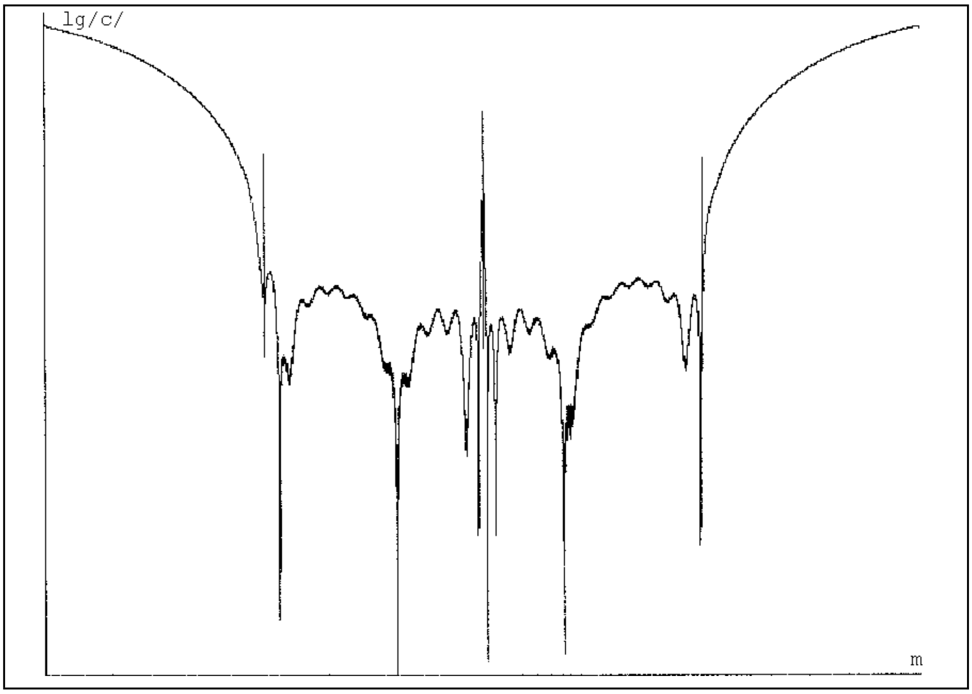

Figure 8 - Complex cepstrum, shown semi-logarithmically

Comment

Once you have calculated a few transformation pairs using the Fourier transform and displayed them graphically, it is easier to recognize the relationships between a time function and its transform. For example, short pulse-like changes in the time function indicate larger values of the transform at the higher frequencies in the spectrum.

The user must be aware that the entire information content of the time function is contained unchanged in its transform. This fact then leads directly to the question: Can components such as interference or noise be isolated from the Fourier transform by elementary mathematical operations? The author also asked himself this question when he was able to learn what he considered to be the ingenious theory of the Fourier transform at the University of Applied Sciences in Krefeld.

One such elementary mathematical operation is the logarithmization of the spectrum, which in the above example of continuous calculation leads to a summand that contains the pulse duration in complex form.

As already indicated, this is a mathematical discontinuity in the time function (gap or kink). One possibility is the superimposition of an echo, whereby the discontinuity in the time function arises at the point in time at which, for example, the transmitted pulse and the received echo coincide. This situation results in the enormous sensitivities of some commercially available RADAR devices, as the author was able to experience for himself when operating a ship's radar (in smooth seas, it was possible to detect a
0.5 nm to locate aluminum beverage cans floating on the water).

As shown in this article, it is also conceivable to determine the time duration up to the point of discontinuity. The pulse length would be determined with approximately the same accuracy as the sampling frequency.

Furthermore, it should be considered to what extent mathematical manipulations in the cepstrum and subsequent back-transformation can contribute to the reconstruction of an interrupted signal.

In addition, one could also think about other elementary mathematical operations applied to the Fourier transform.

Bibliography

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